Math 25

Fall 2017

Lecture 7

$y = ax^2 + bx + c$
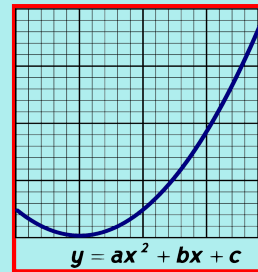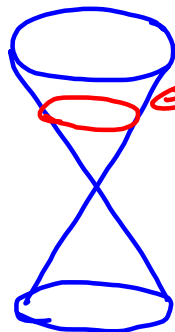

Ch. 7   Analytic Geometry
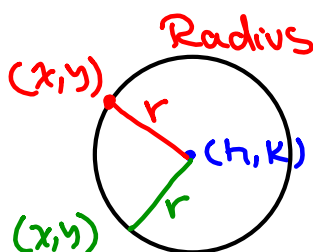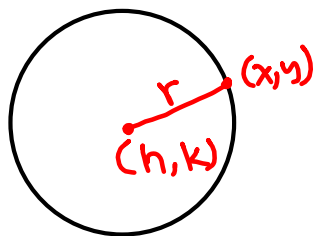
You should know   Conic Sections

Circle

$$(x-h)^2 + (y-k)^2 = r^2$$

The set of all points (x,y) that are same

distance from a fixed

Point (h,k) which is

the center of the circle

Radius

(x,y)

r

(h,k)

r

(x,y)

find equation of a circle that has the center (h, K) with radius r.

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

Square both sides

$$(x-h)^2 + (y-k)^2 = r^2$$

find center and radius of the circle

$$x^2 + y^2 - 6x + 8y + 9 = 0$$
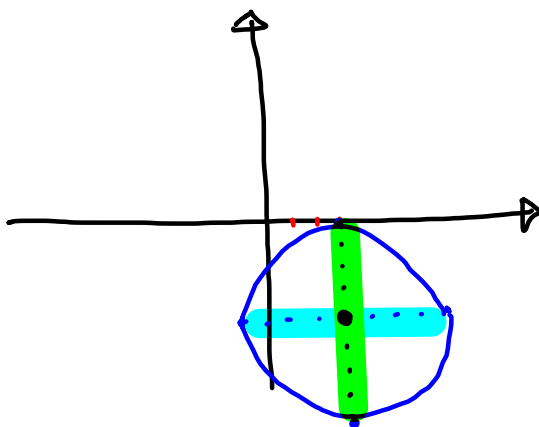
$$x^2 - 6x + 9 + y^2 + 8y + 16 = -9 + 9 + 16$$
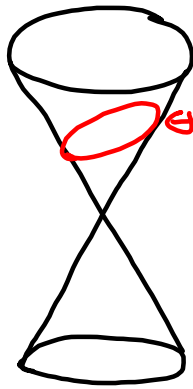
$$(x-3)^2 + (y+4)^2 = 16$$

Center (3, -4)

Radius 4

D: [-1, 7]

R: [-8, 0]

Ellipse

The set of all points that the sum of their distances from two fixed points is constant.

V_2 F_2 (0,0) F_1 V_1
$(-a,0)$ $(-c,0)$ $(c,0)$ $(a,0)$
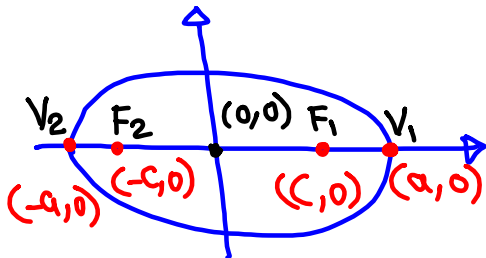
distance from V_1 to F_1 is

$a - c.$

distance from V_1 to f_2 is

$a - (-c) = a + c$

$a + c + a - c = 2a$

V_2 F_2 F_1 V_1
$(-a,0)$ $(-c,0)$ $(c,0)$ $(a,0)$
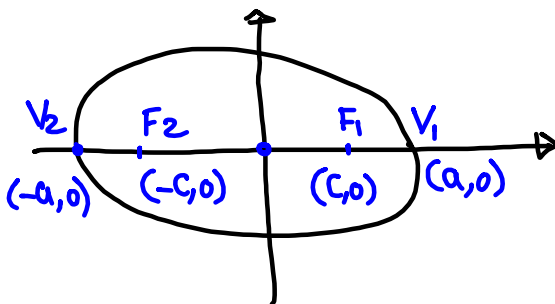
Distance from V_2 to F_1 is $c - (-a) = c + a$

Distance from V_2 to F_2 is $-c - (-a) = -c + a$

Sum of these two distances is $2a$

(x,y)
d_2 d_1
$(-a,0)$ $(-c,0)$ $(c,0)$ $(a,0)$

$d_1 = \sqrt{(x-c)^2 + (y-0)^2}$

$d_2 = \sqrt{(x+c)^2 + (y-0)^2}$

By definition of Ellipse,

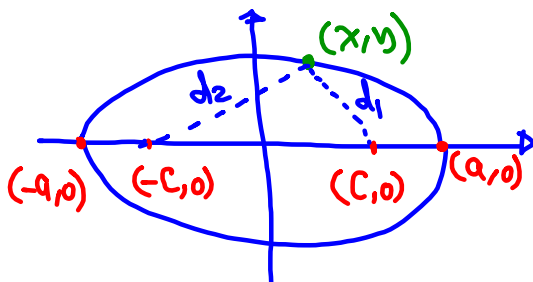$$d_1 + d_2 = 2a$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Square both Sides, $(A-B)^2 = A^2 - 2AB + B^2$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$\cancel{x^2} - 2xc + \cancel{c^2} + \cancel{y^2} = \boxed{4a^2} - 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} + \boxed{2xc} + \cancel{c^2} + \cancel{y^2}$$

$$-2xc \;-4a^2 - 2xc = -4a\sqrt{(x+c)^2 + y^2}$$

$$-4xc - 4a^2 = -4a\sqrt{(x+c)^2 + y^2}$$

Divide by -4

$$xc + a^2 = a\sqrt{(x+c)^2 + y^2}$$

Square both Sides

$$(xc + a^2)^2 = \left(a\sqrt{(x+c)^2 + y^2}\right)^2$$

$$x^2c^2 + 2a^2cx + a^4 = a^2\left[(x+c)^2 + y^2\right]$$

$$\boxed{x^2c^2} + \cancel{2a^2cx} + \boxed{a^4} = a^2x^2 + \cancel{2a^2cx} + a^2c^2 + a^2y^2$$

$$a^2x^2 + a^2c^2 + a^2y^2 - x^2c^2 - a^4 = 0$$

$$(a^2-c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

To make our life easier, Let $b^2 = a^2 - c^2$

$$b^2x^2 + a^2y^2 = a^2b^2$$

Divide everything by a^2b^2

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \quad , \text{Reduce}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b > 0 \qquad \boxed{c^2 = a^2 - b^2}$$

Consider   Ellipse   centered at $(0,0)$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad \begin{array}{ll} a^2 = 25 & a = 5 \\ b^2 = 9 & b = 3 \end{array}$$

Let $y = 0$ $\frac{x^2}{25} = 1 \rightarrow x = \pm 5$

$$c^2 = a^2 - b^2$$
$$= 25 - 9$$

Let $x = 0$ $\frac{y^2}{9} = 1 \rightarrow y = \pm 3$

$$c^2 = 16$$
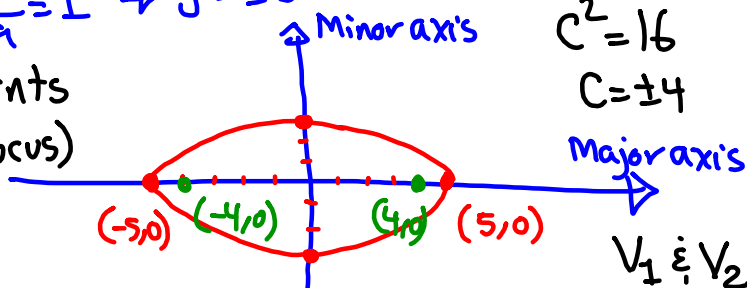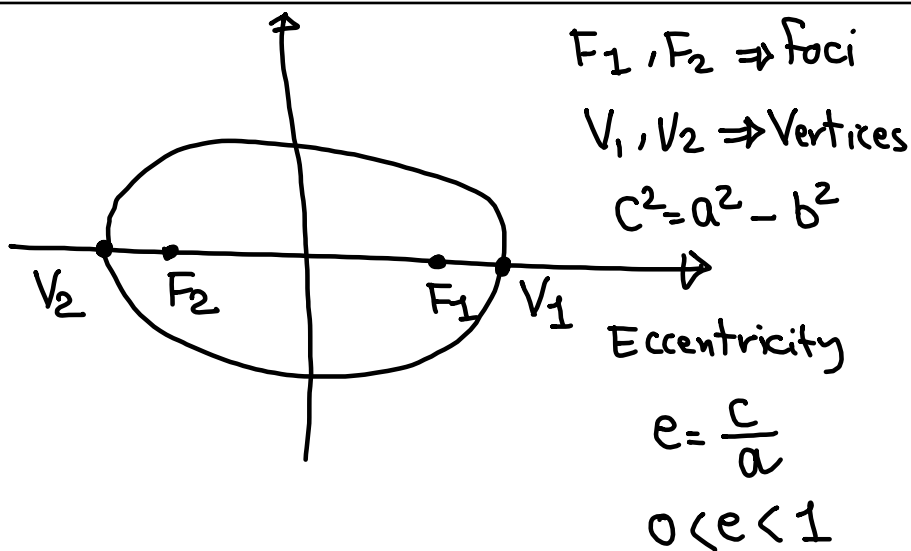$$c = \pm 4$$

Two fixed points
Foci (Plural focus)
$(4,0)$, $(-4,0)$
    F_1 F_2
Two points at the end of major axis are
           called vertices $(5,0),(-5,0)$

Minor axis

Major axis

$V_1 \& V_2$

$(-5,0)$ $(-4,0)$ $(4,0)$ $(5,0)$

$F_1, F_2 \Rightarrow$ foci

$V_1, V_2 \Rightarrow$ Vertices

$C^2 = a^2 - b^2$

Eccentricity

$e = \dfrac{c}{a}$

$0 < e < 1$

Consider the ellipse below

$$\dfrac{y^2}{100} + \dfrac{x^2}{36} = 1$$

Center at $(0,0)$

$a^2 = 100 \rightarrow a = 10$

$b^2 = 36 \rightarrow b = 6$

$C^2 = a^2 - b^2 = 100 - 36 = 64$

$\rightarrow C = 8$

$e = \dfrac{C}{a} = \dfrac{8}{10} = \dfrac{4}{5}$

Major

$V_1 \quad (0,10)$

$F_1 \quad (0,8)$

$(-6,0)$

$(6,0)$

Minor

$F_2 \quad (0,-8)$

$V_2 \quad (0,-10)$
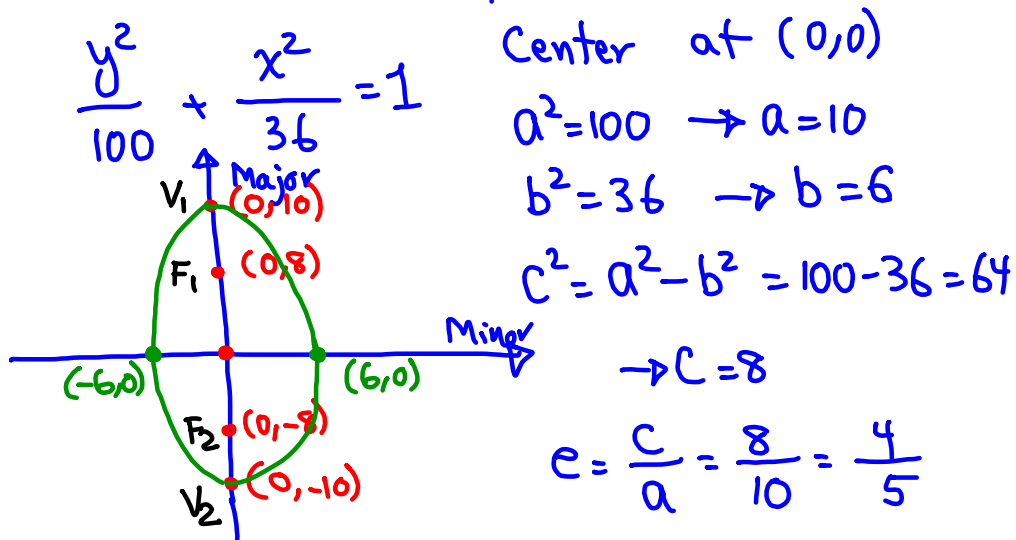
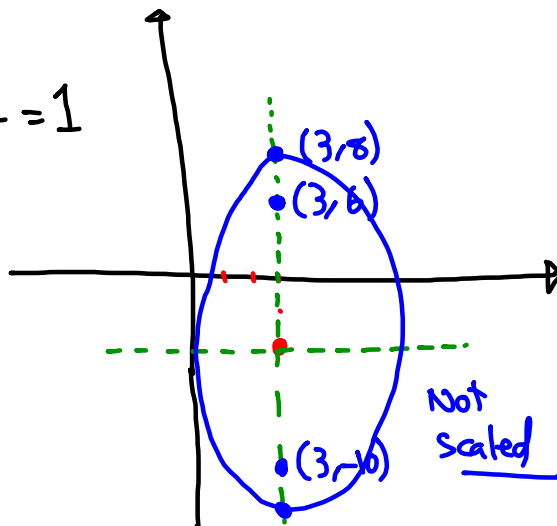I like to move the last ellipse 3 units to the right, and 2 units down.

$$\frac{(y+2)^2}{100} + \frac{(x-3)^2}{36} = 1$$

$(0,8) \rightarrow (3,6)$

$(0,10) \rightarrow (3,8)$

$(0,-8) \rightarrow (3,-10)$

$(0,-10) \rightarrow (3,-12)$

$(3,8)$
$(3,6)$
$(3,-10)$

Not Scaled

Ellipse centered at (h,k)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$a > b > 0$
Major axis → Horizontal

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

Major axis → Vertical

we can adjust accordingly Vertices & foci.

Do a Complete graph:

$$x^2 + 9(y+1)^2 = 81$$

Divide by 81 & reduce, why? RHS = 1

$$\frac{x^2}{81} + \frac{(y+1)^2}{9} = 1$$
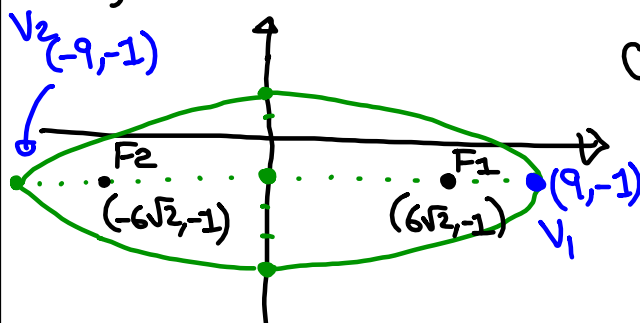
Center $(0, -1)$

$a^2 = 81 \qquad a = 9$

Major axis → Horizontal

$b^2 = 9 \qquad b = 3$

$c^2 = a^2 - b^2 = 81 - 9 = 72$

$c = \sqrt{72} = 6\sqrt{2}$

$e = \dfrac{c}{a} = \dfrac{6\sqrt{2}}{9} = \dfrac{2\sqrt{2}}{3}$

$V_2(-9,-1)$

F_2 $(-6\sqrt{2}, -1)$

F_1 $(6\sqrt{2}, -1)$

$(9,-1)$ V_1

Consider $3x^2 + 2y^2 - 30x - 4y + 59 = 0$

1) Rewrite this eqn in Stand. form of an ellipse

$$\boxed{\frac{(x-h)^2}{a^2}} + \boxed{\frac{(y-k)^2}{b^2}} = 1$$

$a > b > 0$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

$3x^2 - 30x \qquad +2y^2 - 4y \qquad = -59$

$3(x^2 - 10x \ +25\ ) + 2(y^2 - 2y \ +1 \ ) = -59 + 75 + 2$

$$3(x-5)^2 + 2(y-1)^2 = 18$$

Make RHS equal to 1 by dividing by 18.

$$\frac{(x-5)^2}{6} + \frac{(y-1)^2}{9} = 1$$

Center $(5, 1)$

$a^2 = 9$ $a = 3$

$$\frac{(y-1)^2}{9} + \frac{(x-5)^2}{6} = 1$$

$b^2 = 6$ $b = \sqrt{6}$

$c^2 = 3$ $c = \sqrt{3}$

Vertices $(5, 4), (5, -2)$

Foci $(5, 1+\sqrt{3})(5, 1-\sqrt{3})$

Endpoints on minor axis $(5-\sqrt{6}, 1)$

$(5+\sqrt{6}, 1)$

Length of major axis

$2a$ $\boxed{6}$

Length of minor axis

$2b$ $\boxed{2\sqrt{6}}$

Find an equation of ellipse in stand. form with Vertices $(-8, 4)$ & $(2, 4)$ and foci $(-3-\sqrt{21}, 4)$ & $(-3+\sqrt{21}, 4)$.

Center (h, k)

$(-3, 4)$

$(-8, 4)$ $(-3, 4)$ $(2, 4)$

Center

a is the distance from center to Vertex $\boxed{a = 5}$

C is the distance from focus to Center. $\boxed{C = \sqrt{21}}$

$c^2 = a^2 - b^2$

$(\sqrt{21})^2 = 5^2 - b^2$ $b^2 = 4$ $\boxed{b = 2}$
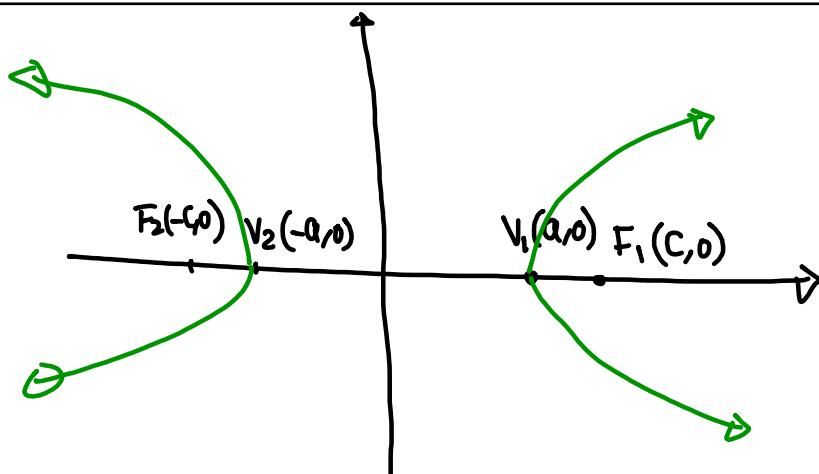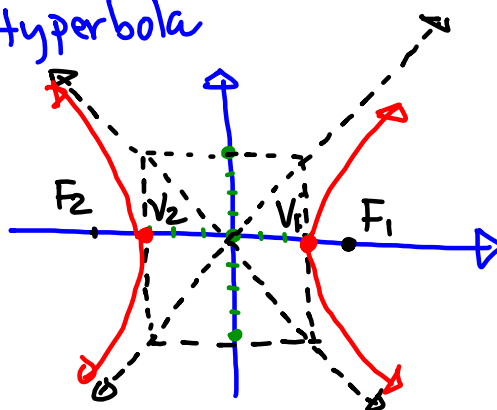
$$\boxed{\frac{(x+3)^2}{25} + \frac{(y-4)^2}{4} = 1}$$

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Recall from Interm. Algebra

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hyperbola

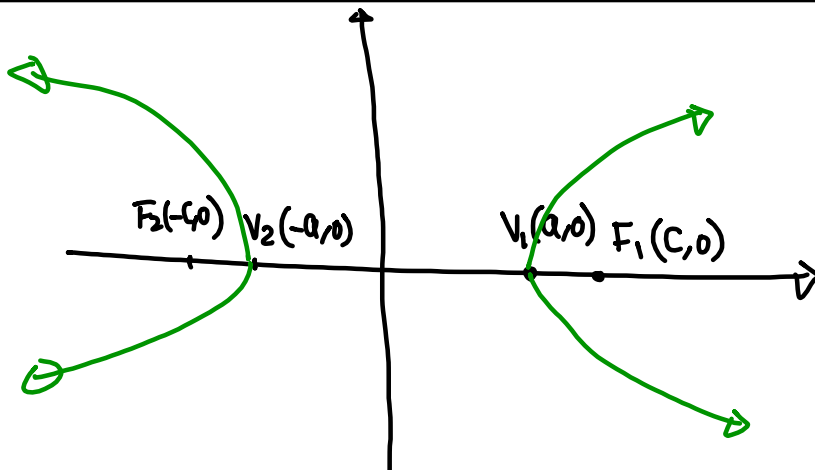$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$d(V_1 \,\&\, F_1) = C - a$$

$$d(V_1 \,\&\, F_2) = C + a$$

$\Rightarrow$ difference of these two

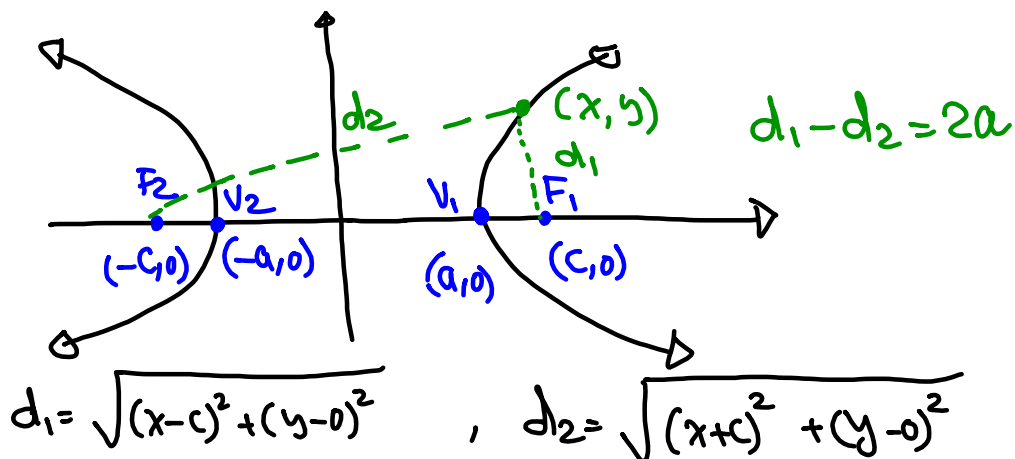distances $\Rightarrow (C + a) - (C - a) = 2a$

$$d(V_2 \,\&\, F_1) = C+a$$

$$d(V_2 \,\&\, F_2) \quad C-a$$

$\Rightarrow$ difference of these two

distances $\Rightarrow (C+a)-(C-a)=2a$

Definition of hyperbola:

The set of all points (x,y) such that the difference of distances from two fixed points (foci) are always the same.

$$d_1 - d_2 = 2a$$

$$d_1 = \sqrt{(x-c)^2 + (y-0)^2} \quad , \quad d_2 = \sqrt{(x+c)^2 + (y-0)^2}$$

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} + 2a$$

Square both sides

using $(A+B)^2 = A^2 + 2AB + B^2$

$$(x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2} + 4a^2$$

$$x^2 + 2xc + c^2 + y^2 = x^2 - 2xc + c^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2} + 4a^2$$

$$2xc + 2xc - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

$$4xc - 4a^2 = 4a\sqrt{(x-c)^2 + y^2}$$

Divide by 4 to reduce

$$xc - a^2 = a\sqrt{(x-c)^2 + y^2}$$

Square both sides again,

$$(xc - a^2)^2 = \left(a\sqrt{(x-c)^2 + y^2}\right)^2$$

$$x^2c^2 - 2ca^2x + a^4 = a^2\left((x-c)^2 + y^2\right)$$

$$x^2c^2 - 2ca^2x + a^4 = a^2x^2 - 2ca^2x + a^2c^2 + a^2y^2$$

$$x^2c^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$$

$a, b > 0$
$a \geq b > 0$

Let $b^2 = c^2 - a^2$

$$b^2 x^2 - a^2 y^2 = a^2 b^2 \quad \text{where}$$

$$b^2 = c^2 - a^2$$

Divide by $a^2 b^2$

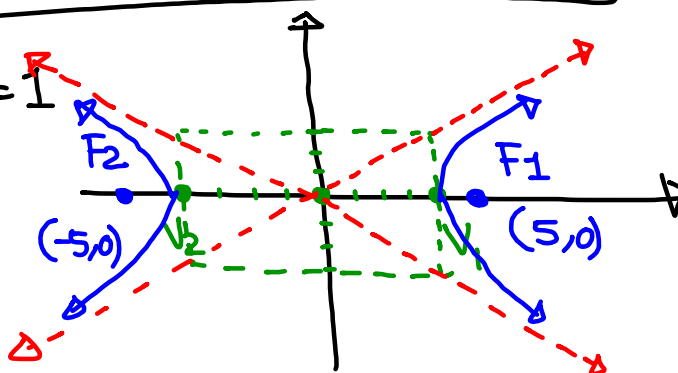$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad , \quad c^2 = a^2 + b^2}$$

Graph $\quad \dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

$C^2 = a^2 + b^2$

$\quad = 16 + 9$

$\quad = 25$

$\boxed{C = 5}$

$F_2 \qquad F_1$

$(-5,0) \quad V_2 \qquad (5,0)$

$$\frac{(y-4)^2}{64} - \frac{(x+2)^2}{36} = 1$$

Hyperbola

opens up & down

Center $(-2,4)$

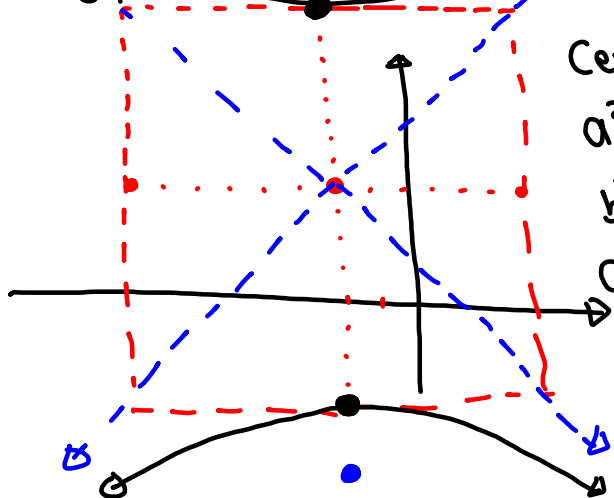$a^2 = 64 \qquad a = 8$

$b^2 = 36 \qquad b = 6$

$C^2 = a^2 + b^2 \qquad C = 10$

$\quad = 100$

$V_1(-2,12), V_2(-2,-4)$

$F_1(-2,14), F_2(-2,-6)$

$$e = \frac{C}{a} = \frac{10}{8} = \frac{5}{4}$$

$e > 1$

$$16(x-4)^2 - 9(y+2)^2 = -144$$

Make RHS 1 : Divide by -144

$$-\frac{(x-4)^2}{9} + \frac{(y+2)^2}{16} = 1 \qquad \frac{(y+2)^2}{16} - \frac{(x-4)^2}{9} = 1$$

Hyperbola, center $(4,-2)$, opens up/down

$$a^2 = 16 \qquad b^2 = 9 \qquad c^2 = 25$$
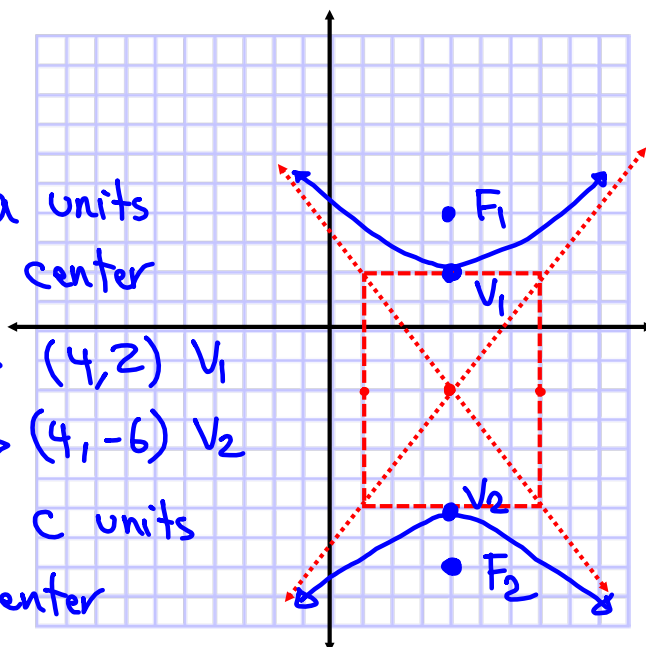$$a = 4 \qquad b = 3 \qquad c = 5$$

Vertices a units from the center

$(4,-2) \rightarrow (4,2) \ V_1$

$\rightarrow (4,-6) \ V_2$

Foci are c units from center

$(4,-2) \rightarrow (4,3)$

$\rightarrow (4,-7)$

Consider $-5x^2 + 9y^2 + 20x - 72y + 79 = 0$

1) write in Stand. form of the hyperbola.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$-5x^2 + 20x \qquad\qquad +9y^2 - 72y \qquad\qquad = -79$

$-5(x^2 - 4x + 4) + 9(y^2 - 8y + 16) = -79 - 20 + 144$
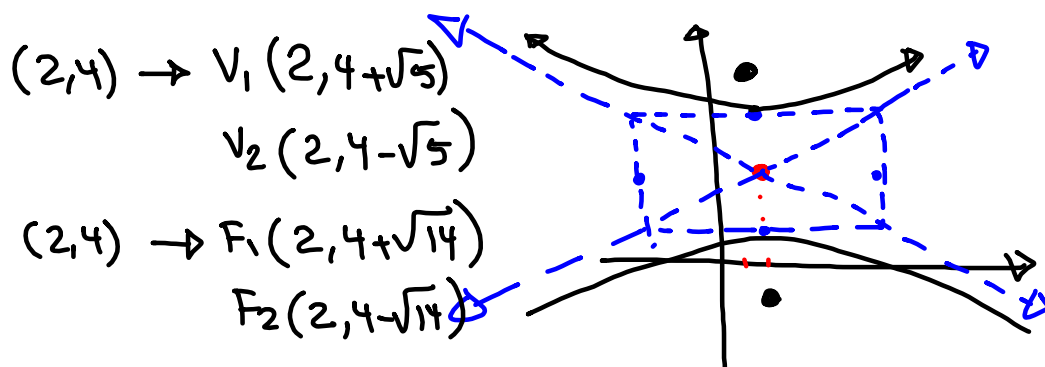
$-5(x-2)^2 + 9(y-4)^2 = 45$

Divide by 45 to make RHS=1

$$\frac{(y-4)^2}{5} - \frac{(x-2)^2}{9} = 1$$

Center $(2,4)$

$a^2 = 5 \qquad b^2 = 9 \qquad c^2 = 5 + 9 = 14$

$a = \sqrt{5} \qquad b = 3 \qquad c = \sqrt{14}$

$(2,4) \rightarrow V_1(2, 4+\sqrt{5})$

$\qquad\qquad V_2(2, 4-\sqrt{5})$

$(2,4) \rightarrow F_1(2, 4+\sqrt{14})$

$\qquad\qquad F_2(2, 4-\sqrt{14})$

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September 30, 2017

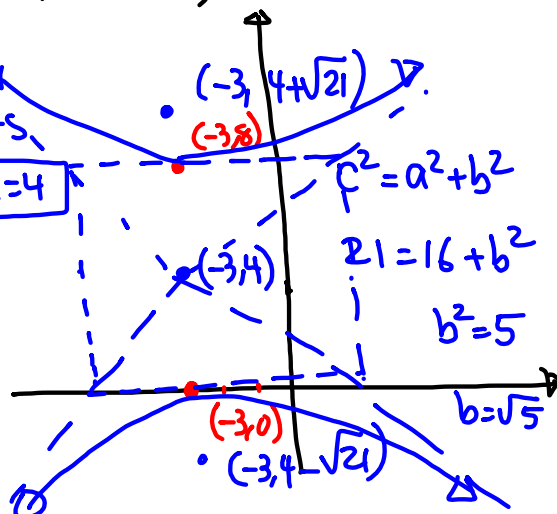Find eqn of a hyperbola in standard form with vertices $(-3,0)$, $(-3,8)$ and foci $(-3, 4+\sqrt{21})$, $(-3, 4-\sqrt{21})$. Draw too.

Center $(-3,4)$

Vertices are a units from the center $\boxed{a=4}$

Foci are c units from the center $\boxed{c=\sqrt{21}}$

$$\frac{(y-4)^2}{16} - \frac{(x+3)^2}{5} = 1$$

$c^2 = a^2 + b^2$

$21 = 16 + b^2$

$b^2 = 5$

$b = \sqrt{5}$

$(-3, 4+\sqrt{21})$
$(-3,8)$
$(-3,4)$
$(-3,0)$
$(-3, 4-\sqrt{21})$
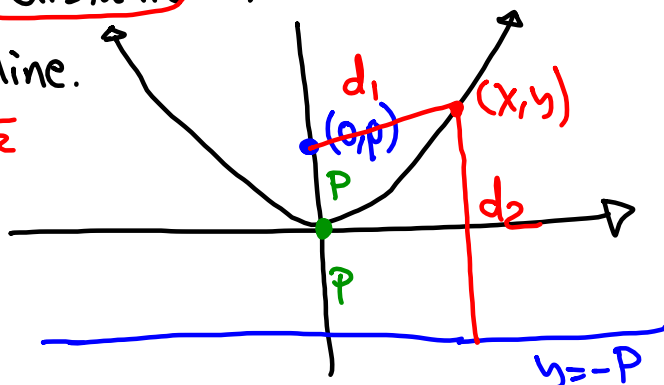
Conic Sections
1) Circle
2) Ellipse
3) Hyperbola
4) Parabola

16

Parabola is the set of all points (x, y) that are (same distance) from a fixed point and a fixed line.

$d_1 = \sqrt{(x-0)^2 + (y-P)^2}$

$d_2 = y + P$

$(0, P)$

d_1

(x, y)

P

d_2

P

$y = -P$

$\sqrt{x^2 + (y-P)^2} = y + P$

$x^2 + (y-P)^2 = (y+P)^2$

$x^2 + y^2 - 2yP + P^2 = y^2 + 2yP + P^2$

$x^2 = 4Py$

Graph $\quad x^2 = 12y$

$x^2 = 4 \cdot 3 \, y$

$\boxed{P = 3}$

$(0, 3)$

Graph

$$(x-5)^2 = 8(y+2)$$

$(0,0) \rightarrow (5,-2)$

$4p = 8 \qquad P = 2$

Focus

2

Vertex

2

Directrix

opens vertically $\rightarrow x^2$

opens horizontally $\rightarrow y^2$

P indicates open up,down, right left

P>0   P<0   P>0   P<0